MATH 2230A

Tutorial 5 12-10-2016

Topics: ∘ Integration over simple closed curve

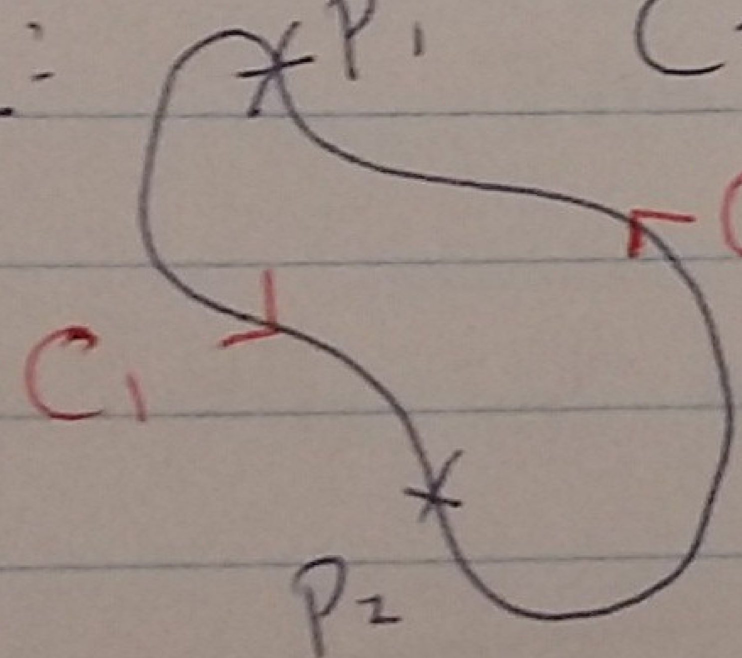
Last time: We already proved that

Anti-derivative exists i.e. $f(z) = F'(z)$ for some F	$\Rightarrow$ Path independence i.e. $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$ whenever $\gamma_1(a) = \gamma_2(a)$ $\gamma_1(b) = \gamma_2(b)$

Now we consider integration over simple closed curve.

Prop: If f is path-independent, then we have $\int_C f = 0$ for any simple closed curve C.

Pf:



$C = C_1 \cup C_2$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$
$$= \int_{C_1} f(z)\,dz - \int_{-C_2} f(z)\,dz$$

Since C_1 and $-C_2$ have the same end points,

$\int_{C_1} f(z)\,dz = \int_{-C_2} f(z)\,dz$ by path independence.

Hence $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{-C_2} f(z)\,dz = 0$

As a result, we have

$$\boxed{\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz \Rightarrow \int_C f(z)\,dz = 0 \ \forall \text{ simple closed curve } C}$$

<u>Example</u>: 1) Consider $f(z)=\frac{1}{z}$, $\gamma(\theta)=Re^{i\theta}$, $\theta\in[0,2\pi]$

Note that
$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(Re^{i\theta})\gamma'(\theta)\,d\theta = \int_0^{2\pi}\frac{1}{Re^{i\theta}}Rie^{i\theta}\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i \neq 0.$$

As a result, $f(z)=\frac{1}{z}$ has no anti-derivative and is not path-independent on the domain $\mathbb{C}\setminus\{0\}$.

<u>Remark</u>: (1) Note that $\operatorname{Log} z$ is <u>not</u> an anti-derivative of $f(z)=\frac{1}{z}$ on $\mathbb{C}\setminus\{0\}$ since it is not continuous on $\mathbb{C}\setminus\{z\}$.

(2) If we consider a smaller domain (e.g. $\{x+iy \mid y>0\}$), then $\operatorname{Log} z$ is an anti-derivative of $f(z)=\frac{1}{z}$.

This shows that we also need to take care of the domain.

Actually we have the following picture:

Anti-derivative exists ($f(z)=F'(z)$) $\xRightarrow[\text{(see last tutorial note)}]{\text{(Essentially Chain Rule)}}$ Path independence ($\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$)

$\Downarrow$ (By prop. in P.1)

$\int_C f(z)\,dz = 0$ $\forall$ simple closed curve C

(Path-independence $\Rightarrow$ $F(z)=\int_p^z f(s)\,ds$ well-defined.)

Q: Can we weaken the condition "f has an anti-derivative"?

Let's recall a theorem from Advanced Calculus:

Thm: (Green's theorem)

- Let C be a simple closed curve in $\mathbb{R}^2$ enclosing a region D.
  Let $f: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (M(x,y), N(x,y))$ be a C^1 function.
  Then we have

$$\oint_C (M\,dx + N\,dy) = \iint_D (N_x - M_y)\,dA$$

Using Green's thm, we can show that

Thm: Let C be a simple closed curve enclosing a region $D \subset \Omega$.
Let $f: \Omega \to \mathbb{C}$ be C^1 and analytic.
Then we have $\int_C f(z)\,dz = 0$ $\forall$ simple closed curve $C \subset \Omega$

Pf: Write $f = u + iv$, $z = x + iy$

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b (u(x,y) + iv(x,y))\cdot(x' + iy')\,dt \\
&= \int_a^b (ux' - vy')\,dt + i\int_a^b (uy' + vx')\,dt \\
&= \int_a^b (u\,dx - v\,dy) + i\int_a^b (u\,dy + v\,dx) \\
&\overset{\text{Green's theorem}}{=} \iint_D (-v_x - u_y)\,dA + i\iint_D (u_x - v_y)\,dA \\
&\overset{\text{By CR-equation}}{=} 0
\end{aligned}$$

$\blacksquare$

Now we have

$$\boxed{f \text{ analytic} + C^1 \implies \int_C f(z)\,dz = 0 \ \forall \text{ simple closed curve } C \text{ enclosing a region } D \subset \Omega}$$

Actually the condition of being C^1 can be dropped.

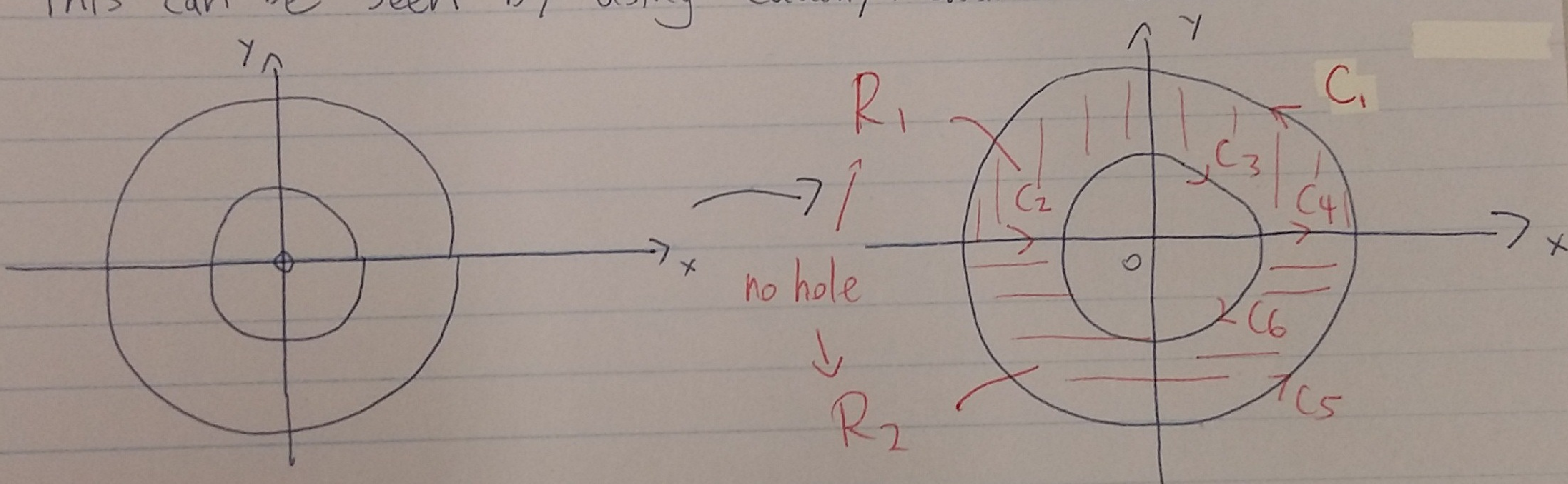
<u>Thm</u>: (Cauchy-Goursat)
- If f is analytic on a simply connected domain Ω, then

$$\int_C f(z)\,dz = 0 \quad \forall \text{ simple closed curve } C \subset \Omega.$$

<u>Remark</u>: As a result, if the domain is $\mathbb{C}$ and $f: \mathbb{C} \to \mathbb{C}$ is entire, then f must be path-indepent and have anti-derivative.

<u>Q</u>: What happens if the domain has 'holes'?

Recall the example $f(z) = \frac{1}{z}$ and $\gamma(\theta) = Re^{i\theta}$, $\theta \in [0, 2\pi]$
Note that $\int_\gamma f(z)\,dz = 2\pi i$ is independent on R.
This can be seen by using Cauchy-Goursat theorem.



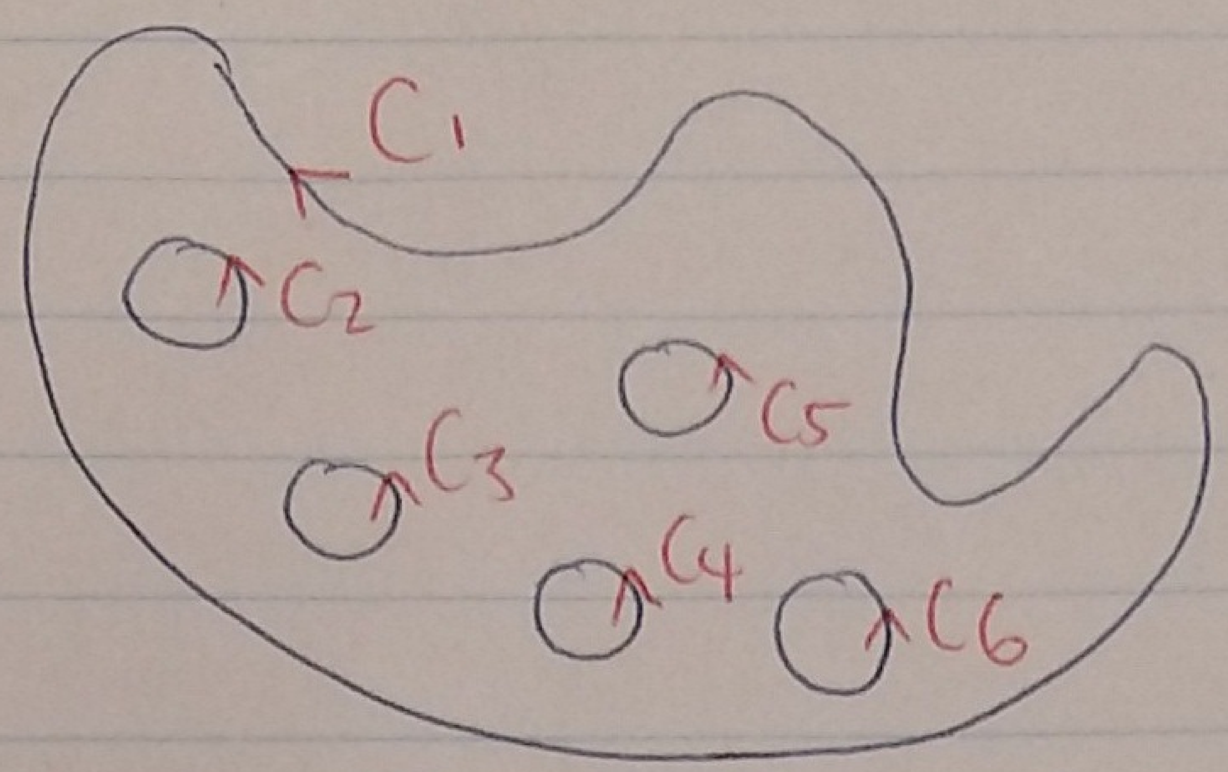
By Cauchy-Goursat theorem,

$$\int_{C_1+C_2+C_3+C_4} f(z)\,dz = 0 = \int_{C_5-C_4+C_6-C_2} f(z)\,dz$$

$$\Rightarrow \int_{C_1+C_5+C_3+C_6} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1+C_5} f(z)\,dz = \int_{-C_3-C_6} f(z)\,dz$$

<u>Remark</u>: This method can be generalized to other domain like



$$\int_{C_1} f(z)\,dz = \sum_{i=2}^{6} \int_{C_i} f(z)\,dz$$